# Phase diagram of the two-dimensional $\pm J$ Ising spin glass

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The $\pm J$ Ising spin glass [probabilities $p$ and $(1-p)$ associated with ferromagnetic and antiferromagnetic couplings, respectively] is studied by applying a real-space renormalization-group technique on a hierarchical lattice that approaches the square lattice. Within such a procedure, there is no spin-glass phase and only two finite-temperature phases are found, namely, the paramagnetic and ferromagnetic ones. In spite of a reasonably small computational effort, an accurate paramagnetic-ferromagnetic boundary is presented: the estimate for the slope at $p=1$ is in very good agreement with the well-known exact result, whereas the coordinates of the Nishimori point are determined within a high precision. Below the Nishimori point, such a boundary is not strictly vertical—contrary to the usual belief—in such a way that a small reentrance is found at low temperatures.

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## I. INTRODUCTION

In spite of a large effort dedicated to the spin-glass (SG) problem [1–3], it still remains as a great challenge in the physics of disordered magnets. Some points are currently accepted as satisfactorily understood; for example, different numerical approaches [4–14], suggest that the nearest-neighbor-interaction Ising SG on a cubic lattice presents a phase transition at finite temperatures. Although some works claim a possible phase transition at finite temperatures [15,16], the majority of the investigations carried out for the same model on a square lattice [4,9–11,14,17–24,26–30] do not find evidence of a SG phase at finite temperatures. Even though the Ising SG on a square lattice may seem a trivial problem (due to the absence of a finite-temperature SG phase), it has attracted the attention of many workers recently, either for its chaotic behavior [22,23] or for its critical behavior at zero-temperature [14,17–20,24,28,30] or along the paramagnetic-ferromagnetic frontier [21,24–27,29].

In the case of a $\pm J$ Ising spin glass, with probabilities $p$ and $(1-p)$ associated with ferromagnetic and antiferromagnetic couplings, respectively, there exists a line along which the internal energy can be calculated exactly; the so-called Nishimori line [31,32] is defined in the plane temperature $T$ vs probability $p$ as

$$\exp(2J/k_BT)=\frac{p}{p-1}. \tag{1}$$

The intersection of the Nishimori line with the border of the ferromagnetic phase is called the Nishimori point [coodinates $(p_N,T_N)$]; for sufficiently high dimensions, in such a way that a SG phase exists at finite temperatures, it has been proposed that the Nishimori point should coincide with the multicritical point where all phases of the model meet [25,33]. In the two-dimensional case, even though the SG phase is absent, the Nishimori point is expected to coincide with a critical point, unstable along the phase boundary, which is probably the simplest critical point appearing in a two-dimensional system at both finite temperature and finite disorder strength [25]. One of the results due to Nishimori is that if one considers a given value of $p$ satisfying Eq. (1), corresponding to a point inside the paramagnetic phase, then the ferromagnetic order parameter should vanish for all temperatures. Such a result implies that, below the Nishimori point, the border of the ferromagnetic line should be either vertical (parallel to the temperature axis), or should bend towards higher values of $p$. If one defines $p_c$ as the value at which such a critical frontier meets the zero-temperature axis, then one should have $p_c \geqslant p_N$. If the inequality holds, then one should have a reentrance within a given range of values of $p$: for the two-dimensional case, by lowering the temperature, one should go from a paramagnetic state to a ferromagnetic one and then back to a paramagnetic phase. However, for sufficiently high dimensions, one comes into a SG phase (which presents a higher entropy as compared with the ferromagnetic one) at low temperatures; the terminology reentrance has been also used in such a case, even though being not accurate, since one does not come back to the same phase by lowering the temperature.

Even though there are some theoretical arguments [34] in favor of the so-called *no-reentrance hypothesis* (supporting a vertical straight line below the Nishimori point, i.e., $p_c = p_N$), there appears to be no fundamental reason why reentrances should be ruled out of thermodynamic systems [35]. Although some numerical investigations [29,36–39] were not able to detect a reentrance in the two-dimensional $\pm J$ Ising spin glass, the possibility of small deviations between $p_c$ and $p_N$ should not be discarded. The most recent finite-temperature numerical approaches yield estimates of the Nishimori point that are very close to one another, e.g., series expansions [21] [$p_N=0.886(3)$], Monte Carlo analysis of nonequilibrium relaxation [27] [$p_N=0.8872(8)$], and numerical transfer matrix [29] [$p_N=0.8905(5)$]. However, such estimates for $p_N$ are slightly smaller than those of $p_c$ obtained through a recent finite-size scaling analysis of exact ground states [24], which find $p_c=0.896(1)$ or $p_c$

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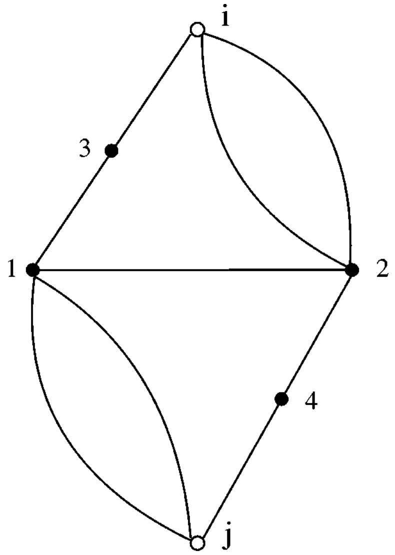


FIG. 1. The basic cell of the hierarchical lattice with fractal dimension $D=2$. The solid circles denote the internal sites (to be decimated in the renormalization process), whereas the open ones represent the external sites (connected to other cells of the lattice).

$=0.894(2)$, depending on the type of scaling employed. The above results suggest $p_c > p_N$, leading to a possible breakdown of the no-reentrance hypothesis.

In the present work we consider the $\pm J$ Ising spin glass on a hierarchical lattice that approaches the square lattice. By applying a renormalization-group approach, we are able to increase the accuracy, with respect to the previous works, in the estimation of the critical points. A precise phase diagram is presented; in particular, it is shown that below the Nishimori point the phase boundary is not strictly vertical. In Sec. II we define the model and the numerical formalism; in Sec. III we present and discuss our results.

## II. THE MODEL AND THE NUMERICAL PROCEDURE

Let us consider the Ising spin glass defined in terms of the Hamiltonian,

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \quad (S_i = \pm 1), \tag{2}$$

where the coupling constants $\{J_{ij}\}$ are quenched random variables following the bimodal (or $\pm J$) probability distribution,

$$P(J_{ij}) = p\,\delta(J_{ij}-J) + (1-p)\,\delta(J_{ij}+J). \tag{3}$$

The sum $\Sigma_{\langle ij \rangle}$ is restricted to nearest-neighbor pairs of spins on a hierarchical lattice generated in such a way that the $(n+1)$th hierarchy is obtained by replacing each single bond of the $n$th hierarchy by a cell like the one exhibited in Fig. 1 [40]; such an operation corresponds to a scaling factor $b=3$. The fractal dimension of the cell in Fig. 1 is $D=\ln 9/\ln 3=2$. Under a renormalization-group (RG) transformation, this cell preserves antiferromagnetism and reproduces several well-known exact results, e.g., the critical temperatures of both pure ferromagnetic and antiferromagnetic Ising models on a square lattice. Such a hierarchical lattice has been employed successfully to approach the square lattice with a large variety of physical systems, such as anisotropic bond percolation, anisotropic Potts ferromagnets, and Ising antiferromagnets [40–44]. Therefore, the model defined through Eq. (1), on the hierarchical lattice defined above, is expected to be a good approximation of the $\pm J$ Ising SG on a square lattice.

The RG procedure may now be carried; as usual, it works inversely to the generation of the hierarchical lattice, i.e., it transforms the cells in Fig. 1 into elementary bonds. Herein, we shall work with the dimensionless exchange energies, $K_{ij} \equiv J_{ij}/k_B T$; the recursion relation involving the effective exchange energy $K'_{ij}$ and the set of original couplings $\{K_{lm}\}$ of a basic cell is given by

$$K'_{ij} = \frac{1}{2} \ln \left[ \frac{\Sigma_{i=1}^{16} \exp(A_i)}{\Sigma_{i=1}^{16} \exp(B_i)} \right], \tag{4}$$

where

$$A_1 = K_{i3} + K_{31} + \tilde{K}_{i2} + K_{12} + \tilde{K}_{1j} + K_{24} + K_{4j},$$

$$A_2 = -K_{i3} - K_{31} + \tilde{K}_{i2} + K_{12} + \tilde{K}_{1j} + K_{24} + K_{4j},$$

$$A_3 = K_{i3} + K_{31} + \tilde{K}_{i2} + K_{12} + \tilde{K}_{1j} - K_{24} - K_{4j},$$

$$A_4 = K_{i3} - K_{31} + \tilde{K}_{i2} - K_{12} - \tilde{K}_{1j} + K_{24} + K_{4j},$$

$$A_5 = K_{i3} + K_{31} - \tilde{K}_{i2} - K_{12} + \tilde{K}_{1j} - K_{24} + K_{4j},$$

$$A_6 = -K_{i3} - K_{31} + \tilde{K}_{i2} + K_{12} + \tilde{K}_{1j} - K_{24} - K_{4j},$$

$$A_7 = -K_{i3} + K_{31} + \tilde{K}_{i2} - K_{12} - \tilde{K}_{1j} + K_{24} + K_{4j},$$

$$A_8 = -K_{i3} - K_{31} - \tilde{K}_{i2} - K_{12} + \tilde{K}_{1j} - K_{24} + K_{4j},$$

$$A_9 = K_{i3} - K_{31} + \tilde{K}_{i2} - K_{12} - \tilde{K}_{1j} - K_{24} - K_{4j},$$

$$A_{10} = K_{i3} + K_{31} - \tilde{K}_{i2} - K_{12} + \tilde{K}_{1j} + K_{24} - K_{4j},$$

$$A_{11} = K_{i3} - K_{31} - \tilde{K}_{i2} + K_{12} - \tilde{K}_{1j} - K_{24} + K_{4j},$$

$$A_{12} = K_{i3} - K_{31} - \tilde{K}_{i2} + K_{12} - \tilde{K}_{1j} + K_{24} - K_{4j},$$

$$A_{13} = -K_{i3} + K_{31} - \tilde{K}_{i2} + K_{12} - \tilde{K}_{1j} - K_{24} + K_{4j},$$

$$A_{14} = -K_{i3} - K_{31} - \tilde{K}_{i2} - K_{12} + \tilde{K}_{1j} + K_{24} - K_{4j},$$

$$A_{15} = -K_{i3} + K_{31} + \tilde{K}_{i2} - K_{12} - \tilde{K}_{1j} - K_{24} - K_{4j},$$

$$A_{16} = -K_{i3} + K_{31} - \tilde{K}_{i2} + K_{12} - \tilde{K}_{1j} + K_{24} - K_{4j},$$

with $\tilde{K}_{i2}$ and $\tilde{K}_{1j}$ representing effective exchange energies (i.e., the sum of the exchange energies associated with each of the two parallel paths connecting sites $i$ to 2 and 1 to $j$, respectively). The $B_i$'s ($i=1,...,16$) may be obtained from the respective $A_i$'s by inverting the signs preceding $\tilde{K}_{1j}$ and $K_{4j}$.

At zero temperature, the recursion relation involves the effective coupling $J'_{ij}$ and the set of original couplings $\{J_{lm}\}$ in such a way that

$$J'_{ij}=\tfrac{1}{2}(C_{\max}-D_{\max}), \tag{5}$$

where

$$C_{\max}=\max(C_1,C_2,\ldots,C_{16}),$$

$$D_{\max}=\max(D_1,D_2,\ldots,D_{16}),$$

$$C_i=\lim_{T\to 0} k_BT(A_i); \quad D_i=\lim_{T\to 0} k_BT(B_i); \quad (i=1,...,16). \tag{6}$$

In order to find the phases of the model, one should follow numerically the probability distribution associated with the exchange energies [4]. Such a probability distribution is mimicked by a set of $M$ real numbers $\{K_l^{(n)}\}$, from which one may compute at each iteration step $n$,

$$\sigma_m^{(n)}=\frac{1}{M}\sum_{l=1}^{M}(K_l^{(n)})^m \quad (m=1,2,...), \tag{7}$$

which are expected to approach, in the limit $M\to\infty$, the moments of the distribution $P(K_{ij})$. In order to reduce the dependence of $\sigma_m^{(n)}$ in the particular sequence of random numbers, such quantities were computed, for each RG step, over $N_s$ different samples (different sequences of random numbers), after which, the averages over samples were considered, $[\sigma_m^{(n)}]_{av}$.

For given values of temperature $T$ and probability $p$, the RG process starts by creating an initial pool with $M$ real numbers $\{K_l^{(0)}\}$ produced according to a bimodal probability distribution similar to the one in Eq. (3). An iteration consists in $M$ operations, where in each of them one picks randomly nine numbers from the pool (each chosen number is assigned to a bond in the cell of Fig. 1) in order to generate the corresponding effective coupling according to Eq. (4). After that, one gets a new pool representing the renormalized probability distribution, from which one may compute the moments in Eq. (7). Such moments are stored for each iteration step $n$, and each individual sample, in such a way that $[\sigma_m^{(n)}]_{av}$ may be calculated.

It should be mentioned that the bimodal distribution presents a rapid proliferation of delta functions under the RG process [14]; inside the paramagnetic phase, the delta function at $K_{ij}=0$ (which usually appears throughout the RG procedure) increases its weight after each renormalization, whereas inside the ferromagnetic phase the deltas corresponding to $K_{ij}>0$ increase their weight under the RG process. The SG phase is usually associated with a decrease in the weight of the delta at the origin, whereas the deltas for both positive and negative $K_{ij}$ predominate. Therefore, under the RG process the averaged moments $[\sigma_m^{(n)}]_{av}$ should approach, for increasing values of $n$, $[\sigma_m^{(n)}]_{av}\to 0$ in the paramagnetic phase, $[\sigma_m^{(n)}]_{av}\to\infty$ in the ferromagnetic phase, whereas $[\sigma_m^{(n)}]_{av}\to 0$ ($m$ odd) and $[\sigma_m^{(n)}]_{av}\to\infty$ ($m$ even) in the SG phase.

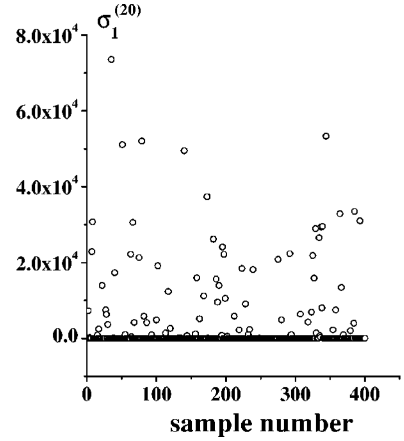


FIG. 2. The first moment of the distribution of exchange energies after 20 RG iterations, for each single sample, for a typical point close to the critical frontier ($p=0.98$, $k_BT/J=2.1175$). The majority of samples (78.75%) presented a convergence to the paramagnetic attractor, whereas a small fraction of samples (21.25%) presented a convergence to the ferromagnetic attractor. Large square-root deviations with respect to $[\sigma_m^{(n)}]_{av}$, $\delta_m^{(n)}$, are observed; in this case, one finds that the ratio $\delta_1^{(n)}/[\sigma_1^{(n)}]_{av}$ remains essentially unchanged (approximately 0.15) for iteration numbers $n\geqslant 15$.

For a more reliable identification of a given phase boundary, one should also monitor the square-root deviations associated with each $[\sigma_m^{(n)}]_{av}$ (herein denoted by $\delta_m^{(n)}$); large values of $\delta_m^{(n)}$ may be an indication that only a small number of samples are contributing significantly to the sample average in $[\sigma_m^{(n)}]_{av}$. As an example, we show in Fig. 2 a typical case, with the sample-to-sample fluctuations of $\sigma_1^{(20)}$ for the present model, in the neighborhood of the critical frontier paramagnetic/ferromagnetic ($p=0.98$; $k_BT/J=2.1175$). For the particular case exhibited in Fig. 2, the fact that $[\sigma_m^{(n)}]_{av}$ increase under successive RG iterations is due to a convergence to the ferromagnetic attractor of a small fraction of samples; the great majority of samples (nearly 80%) present the moments $\sigma_m^{(n)}$ of Eq. (7) with a convergence to a paramagnetic attractor. In cases like that, the corresponding point under study will, herein, be considered inside the paramagnetic phase, instead of in the ferromagnetic one.

### III. RESULTS AND DISCUSSION

We have considered pools of size $M=800\,000$ and our simulations were repeated for $N_s=400$ samples. Within the present numerical approach, we found no evidence of a SG phase at finite temperatures, and only the paramagnetic and ferromagnetic phases were observed; the results that follow refer to the critical frontier separating such two phases. In most of the cases, in order to achieve a proper convergence to one of the attractors, 20 RG iterations were enough, although in some cases, up to 30 RG iterations were necessary.

TABLE I. Estimated critical temperatures of the two-dimensional $\pm J$ Ising spin glass for several values of $p$, above the Nishimori point.

| $p$ | $k_B T_c(p)/J$ |
|---|---|
| 0.995 | $2.2318 \pm 0.0005$ |
| 0.990 | $2.1941 \pm 0.0008$ |
| 0.980 | $2.1165 \pm 0.0011$ |
| 0.970 | $2.0367 \pm 0.0015$ |
| 0.950 | $1.8674 \pm 0.0020$ |
| 0.930 | $1.6790 \pm 0.0025$ |
| 0.910 | $1.4500 \pm 0.0035$ |

For a specific sample, a point in the ($p$, $T$) plane was considered inside the paramagnetic (ferromagnetic) phase if, at the end of the RG process, $\sigma_1^{(n)} < 10^{-3}$ ($\sigma_1^{(n)} > 10$). Let us now define $\eta_P$ ($\eta_F$) as the fraction of total number of samples which, at the end of the RG process, have converged to the paramagnetic (ferromagnetic) attractor. We have associated a point in the ($p$, $T$) plane with a specific phase only when the corresponding fraction of samples, as defined above, was greater than 0.8; this ensures small ratios $\delta_m^{(n)}/[\sigma_m^{(n)}]_{av}$ [e.g., for the case $m=1$, a point with $\eta_F > 0.8$, presents $\delta_1^{(n)}/[\sigma_1^{(n)}]_{av}$ slightly constant for $n > 15$, fixed to a value not greater than 0.05). The uncertainties in our critical-point estimates (see Table I) correspond to situations where $0 < \eta_P,\ \eta_F < 0.8$.

As mentioned before, the hierarchical lattice considered reproduces the exact critical temperature of the two-dimensional ferromagnetic Ising model; in Table I we display typical values of critical temperatures found for different values of $p_N < p < 1$. A good test for the present approach is to compute the reduced slope of the critical frontier at $p = 1$; considering the exact critical temperature at $p=1$, together with the data for $p=0.995$ and $p=0.99$ of Table I, one finds

$$s = \frac{1}{T_c(1)} \left. \frac{dT_c(p)}{dp} \right|_{p=1} = 3.23 \pm 0.03, \qquad (8)$$

which compares rather well with the exact result, $s = 2\sqrt{2}/[\ln(\sqrt{2}+1)] \approx 3.209$ [45].

In order to compute the Nishimori point, we have applied the above-mentioned RG procedure along the Nishimori line [Eq. (1)]; our estimate is [see Fig. 3(a)]

$$p_N = 0.8902 \pm 0.0004, \quad \frac{k_B T_N}{J} = 0.9557 \pm 0.0018, \qquad (9)$$

which is in good agreement with the most recent finite-temperature numerical approaches. Indeed, our estimate agrees well with the recent numerical transfer-matrix approach [29] [$p_N = 0.8905(5)$], but is slightly higher than those obtained through series expansions [21] [$p_N = 0.886(3)$] and Monte Carlo analysis of nonequilibrium relaxation [27] [$p_N = 0.8872(8)$].

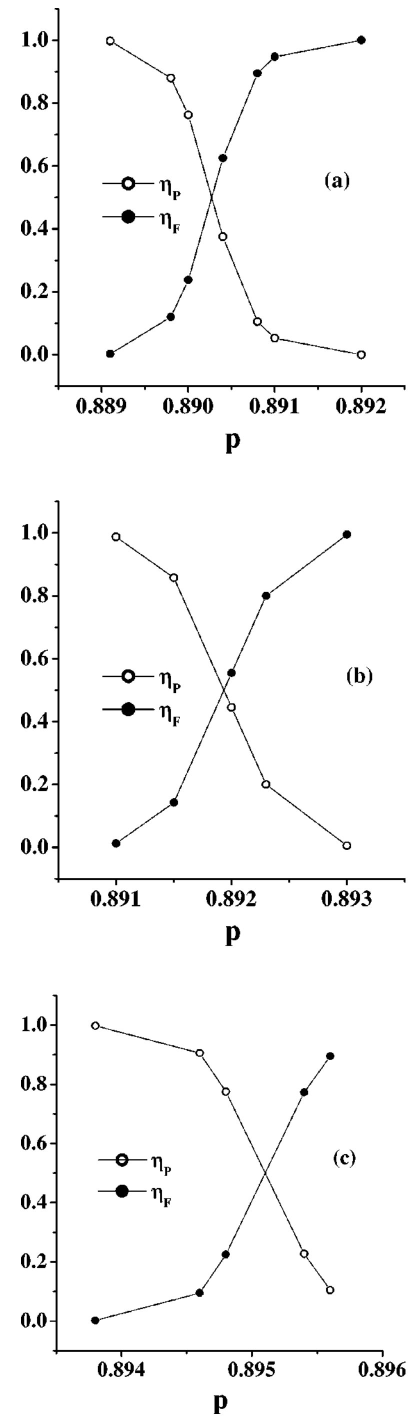


FIG. 3. The fraction of the total number of samples which, at the end of the RG process, have converged to a paramagnetic ($\eta_P$) or ferromagnetic ($\eta_F$) attractor, as function of $p$. The critical points are defined by $\eta_P = \eta_F$ and our uncertainty regions were considered for $0 < \eta_P,\ \eta_F < 0.8$. (a) Along the Nishimori line (estimated Nishimori point: $p_N = 0.8902 \pm 0.0004$); (b) Temperature $k_B T/J = 0.5$ (estimated $p = 0.8919 \pm 0.0004$); (c) Zero temperature (estimated $p_c = 0.8951 \pm 0.0003$). In all cases above, 20 iterations were considered in the RG process.

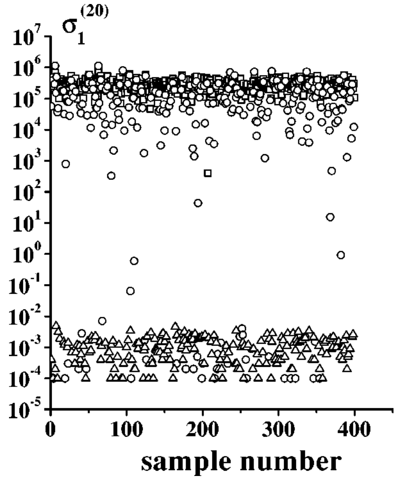


FIG. 4. The first moment of the distribution of exchange energies after 20 RG iterations, for each single sample, at $p=0.892$ and three different temperatures. (a) Along the Nishimori line (squares); (b) $k_BT/J=0.5$ (circles); (c) Zero temperature (triangles). For a better visualization, the samples with $\sigma_1^{(20)}<10^{-5}$ were not represented.

Let us now turn to the critical frontier for temperatures lower than $T_N$. In Fig. 3 one clearly sees that the fractions of samples $\eta_P$ and $\eta_F$ indicate a critical frontier bending towards values of $p$ greater than $p_N$. In Fig. 3(a) one has $\eta_P$ and $\eta_F$ along the Nishimori line, in the neighborhood of the Nishimori point, leading to the estimate of Eq. (9). In Fig. 3(b) one has such fractions for $k_BT/J=0.5$, yielding ($p=0.8919\pm0.0004$, $k_BT_c/J=0.5$). From Fig. 3(c) one gets the critical point at zero temperature,

$$p_c=0.8951\pm0.0003. \tag{10}$$

The above result agrees well with a recent finite-size scaling analysis of exact ground states [24], which obtain $p_c=0.896(1)$ or $p_c=0.894(2)$, depending on the type of scaling employed. Indeed, the result of Eq. (10) lies in between such two estimates.

We have found that $p_c>p_N$, i.e., a small reentrance is observed below the Nishimori point. In order to illustrate this effect more clearly, in Fig. 4 we exhibit the sample-to-sample fluctuations of $\sigma_1^{(20)}$ for a conveniently chosen value of $p$ $(p=0.892)$, at three different temperatures. Along the Nishimori line most of the samples have converged to the ferromagnetic attractor, for $k_BT/J=0.5$ one is clearly very close to the critical frontier, whereas at zero temperature, most of the samples have converged to the paramagnetic attractor. As far as we know, this is the first time that such a reentrance has been observed in the present model. Due to its small extent, in fact smaller than the error bars in some of the previous numerical investigations, it might have been indiscernible up to the moment.

The phase diagram, obtained through the present numerical approach, for the two-dimensional $\pm J$ Ising SG, is shown in Fig. 5; for the sake of clarity, we exhibit only the range $0.7\leqslant p\leqslant1.0$. Since this model holds the symmetry $(p,J)\leftrightarrow(1-p,-J)$, and the present RG approach preserves antiferromagnetism, one has a symmetric phase diagram with respect to the axis $p=1/2$, with an antiferromagnetic phase appearing for low values of $p$.

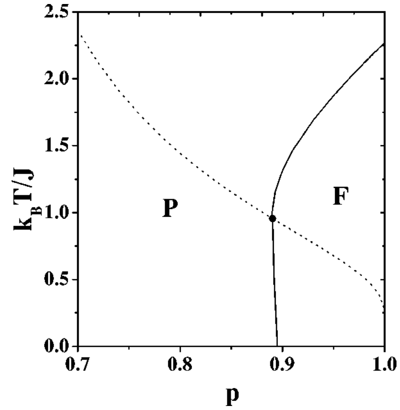


FIG. 5. The phase diagram of the two-dimensional $\pm J$ Ising spin glass, as obtained by the present approach. Only two phases are present, namely, the paramagnetic ($P$) and the ferromagnetic ($F$) ones. The dashed line represents the Nishimori line [Eq. (1)] and the black circle denotes the Nishimori point.

It should be pointed out that the present method, although being very suitable for investigating critical frontiers, is not appropriate for studying the critical exponents of the model considered herein. Under successive RG iterations, the coupling probability distribution changes rapidly, leading to a proliferation of delta functions; this certainly yields critical exponents that are very different from the true critical exponents of the $\pm J$ Ising spin glass on a square lattice.

Finally, we have studied the $\pm J$ Ising spin glass on a hierarchical lattice that approaches the square lattice. A renormalization-group method is employed, and the evolution of the probability distribution associated with the coupling constants is analyzed numerically. Within such a procedure, one is able to obtain a precise paramagnetic-ferromagnetic critical frontier, improving the accuracy with respect to previous investigations. A small reentrance is observed in the critical frontier for temperatures below the Nishimori point. There is always a possibility that such a reentrance may be a peculiarity of the particular hierarchical lattice investigated. However, taking into account the accuracy of the results, either near $p=1$ or the good agreement of the location of the Nishimori point, as well as the zero-temperature critical point, with the most recent numerical investigations, it is very probable that the reentrance found herein is also present in the $\pm J$ Ising spin glass on a square lattice. Further numerical investigations of this model on a square lattice are necessary to clarify this issue.

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